Fields

We begin by defining what a **field** actually is.

Definition 0.1 (Field) — A region of space where every point has an assigned value.

This **assigned value** is simply some number associated with each point in the field. It could be:

1. A **scalar** *e.g.* height, energy.
2. A **vector** *e.g.* force, flow.
3. A **tensor** *e.g.* general relativity.

In the context of physics, this value is often the force applied to some physical quantity.

Definition 0.2 (Uniform Field) — The direction and strength of the field is constant throughout the considered region of space.

Definition 0.3 (Equipotentials) — Lines of constant potential.

For **spherically symmetric mass distributions**, equipotentials are concentric circles.

§0.1 Gravitational Fields

We use the concept of a gravitational field to attempt to answer questions such as

- how do masses in empty space interact?
- how does one mass know that the other is there?
- what goes on in the space between them?

Two masses do not interact directly with one another but instead with the gravitational field established by the other.

The mass of an object, say m_1, creates a gravitational field around it. When mass m_2 is placed in this field, it experiences a force of attraction towards m_1. Of course, m_2 also has a gravitational field that exerts a force on m_1. The two forces are equal in magnitude but opposite in direction; they are a Newton's Third Law pair of forces.

§0.1.1 Newton's Law of Gravitation

> *Every particle in the universe attracts every other particle with a force that is directly proportional to the product of the two masses and inversely proportional to the square of the distance between them.*

Newton's law of gravitation states that the magnitude of the attractive force between two **point particles** of mass m_1 and m_2 separated by a distance r is

Formula 0.4 (Newton's Law of Gravitation)

$$F_g = \frac{Gm_1m_2}{r^2}$$

where G is the fundamental **gravitational constant** $(= 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}})$.
Newton's law of gravitation applies equally well to spherically symmetric mass distributions such as planets.

§0.1.2 Gravitational Field Strength

The gravitational field $\vec{\mathbf{g}}$ is defined as

Definition 0.5 (Gravitational Field Strength) — The gravitational force per unit mass on a small 'test mass' placed in the field.

Formula 0.6 (Gravitational Field)

$$\vec{\mathbf{g}} = \frac{\vec{\mathbf{F}}}{m}$$

Essentially, we imagine releasing the test particle of mass m into the gravitational field and measuring the force acting on it at a particular point in space.[1]
The gravitational field **strength** is the magnitude of the gravitational field $\vec{\mathbf{g}}$. We know

$$\vec{\mathbf{F}} = -\frac{GMm}{r^2}\,\hat{r}\,, \quad \vec{\mathbf{g}} = \frac{\vec{\mathbf{F}}}{m}$$

$$\Rightarrow \vec{\mathbf{g}} = -\frac{GM}{r^2}\,\hat{\mathbf{r}}$$

where $\hat{r}$ is a unit vector in the radial direction. So, gravitational field strength $g = |\vec{\mathbf{g}}|$, which gives

Formula 0.7 (Gravitational Field Strength)

$$g = \frac{GM}{r^2}$$

Example 0.8 Show that the density of the Earth is given by

$$\rho_E = \frac{3g}{4\pi r_E G}$$

[1] The test mass should be sufficiently small so that it does not disturb the body whose gravitational field strength we are measuring.

Solution. We know that $\rho = \frac{M}{V}$ and $g = \frac{GM}{r^2}$.

$$\therefore \ \rho_E = \frac{gr_E^2}{G} \times \frac{1}{\frac{4}{3}\pi r_E^3} = \frac{3g}{4\pi r_E G}$$

□

The path that a 'test mass' follows in a gravitational field is called a **field line**. For a spherical mass distribution the field lines are directed towards the centre of the sphere creating a **radial field**.

In a **uniform field**, both the direction and density of field lines are constant. For **small** changes in height close to the surface, the Earth's gravitational field is approximately uniform.

§0.1.3 Gravitational Potential and GPE

Definition 0.9 (Gravitational Potential Energy) — The work done in moving a mass from infinity to a point in a gravitational field.

Definition 0.10 (Graviational Potential) — The work done per unit mass in moving a mass from infinity to a point in a gravitational field.

Remark 0.11 Unfortunately, those are just the typical, hand-wavy definitions required for A-Level. Below, I present a more mathematically rigorous argument.

The definition of potential energy U is $U(X) = -\int_{x_0}^{X} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{x}}$.

$$\begin{aligned} \Delta U &= -\int_{\infty}^{R} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} \\ &= -\int_{\infty}^{R} -\frac{GMm}{r^2}\mathrm{d}r \\ &= GMm \int_{\infty}^{R} \frac{\mathrm{d}r}{r^2} \\ &= GMm \left[-\frac{1}{r}\right]_{\infty}^{R} \\ &= -\frac{GMm}{r} \end{aligned}$$

This leads itself to the gravitational potential also which is simply the potential energy per unit mass of the 'test mass'.

Formula 0.12 (Gravitational Potential Energy)

$$U = -\frac{GMm}{r}$$

Formula 0.13 (Gravitational Potential)

$$V = \frac{U}{m} = -\frac{GM}{r}$$

Example 0.14 Show that the expression for ΔU reduces to $\Delta U = mg\Delta y$ for small changes in height.

Solution.

$$\begin{aligned}\Delta U &= U(r+y) - U(r)\\ &= -GMm\left(\frac{1}{r+y} - \frac{1}{r}\right)\\ &= -\frac{GM}{r}m\left(\frac{1}{1+\frac{y}{r}} - 1\right)\\ &= \frac{GM}{r}m\left(\frac{\frac{y}{r}}{1+\frac{y}{r}}\right) = \frac{GM}{r^2}my\left(\frac{1}{1+\frac{y}{r}}\right)\end{aligned}$$

But, $\frac{h}{r} \to 0$, so the bracketed expression tends to 1, hence $\Delta U = mg\Delta y$ for small y. □

Close to the Earth's surface, the work done by the gravitational field W_g is given by

$$W_g = \int_{y_i}^{y_f} (-mg)\mathrm{d}y = -mg\Delta y$$

Note that positive y is upwards, and the weight mg acts downwards hence the negative sign.
We call the gravitational potential energy U where we know $U = mg\Delta y$. Thus, the work done by gravity is $W_g = -\Delta U$.

§0.1.4 Escape Velocity

The escape 'velocity' is the initial speed required to go from a point in a gravitational field to infinity with a residual velocity of zero.
By the conservation of energy, inital KE + initial PE = final KE + final PE.

$$\frac{1}{2}mv_e^2 + \left(-\frac{GMm}{r}\right) = 0 + 0$$

We can rearrange this for the escape velocity v_e.

Formula 0.15 (Escape Velocity)

$$v_e = \sqrt{\frac{2GM}{r}}$$

Example 0.16 Calculate the escape velocity at the surface of the Earth.

Solution. We use the facts that $g = 9.81\ \mathrm{N\,kg^{-1}}$ and $R_E = 6400$ km.

$$g = \frac{GM}{r^2} \Rightarrow \frac{GM}{r} = gr$$

$$\therefore\ v_e = \sqrt{2gR_E} \approx \underline{11.2\ \mathrm{km\,s^{-1}}}$$

□

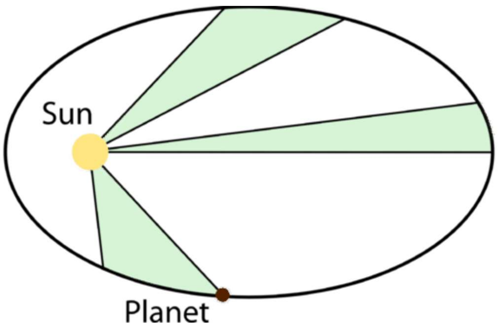
§0.1.5 Kepler's Laws

Kepler's Laws are a set of three laws describing the orbits of planets around the sun. The three laws state:

1. The *law of orbits*: All planets move in **elliptical**[2] orbits, with the sun at one **focus**.
2. The *law of areas*: A line segment joining a planet and the sun sweeps out equal areas during equal time intervals.
3. The *law of periods*: The square of the period of any planet is proportional to the cube of the **semimajor axis** of its orbit (which can be approximated as the radius of the approximate circle).

> **Remark 0.17** While Kepler's first law refers to **elliptical** orbits, these orbits are not very highly elliptical so it is often safe to approximate orbits of planets around the sun as circular.

For greater clarity with the second law, the following illustration is helpful. The green areas are all equal and swept out in equal amounts of time.



We can prove Kepler's third law as follows:
For a planet of mass m moving in a circular orbit of radius r about a star of mass M, by Newton's law of gravitation we know

$$F_g = \frac{GmM}{r^2}$$

but also since this a circular orbit, we have a centripetal force at play.

$$F_c = m\omega^2 r$$

As the gravitational force acts toward the centre of the circular orbit, $F_g = F_c$.

$$\frac{GmM}{r^2} = m\omega^2 r \Rightarrow GM = \omega^2 r^3$$

But, recall that $\omega = \frac{2\pi}{T}$, which gives us

$$GM = \frac{4\pi^2}{T^2} r^3 \Rightarrow T^2 = \frac{4\pi^2}{GM} \cdot r^3$$

Hence, $T^2 \ \alpha \ r^3$.

[2] An ellipse is defined by the locus (or set of points) for which the sum of its distances from the two points called foci add up to a constant value.

§0.2 Electric Fields

§0.2.1 The Electric Field

It is first worth defining what this abstract concept of **charge** actually is.

Definition 0.18 (Charge) — Some conserved quantity in the universe that just so happens to exert forces on itself.

From a mathematical standpoint, we may see charge defined as

$$q = \int_0^T I \cdot \mathrm{d}t$$

The electric field ($\vec{\mathbf{E}}$) is defined as

Definition 0.19 (Electric Field) — The electrostatic force per unit positive test charge applied to said charge.

It is a vector field and multiple electric fields add through vector addition.

Formula 0.20 (Electric Field)

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q}$$

where $\vec{\mathbf{E}}$ is the electric field (NC^{-1}), q is the charge inside the field (C) and ($\vec{\mathbf{F}}$) is the force applied to the charge (N).

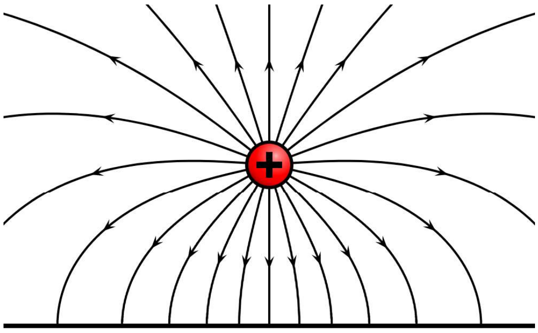
The electric field bears many similarities to the gravitational field. For instance $\vec{\mathbf{E}}$ is a lot like $\vec{\mathbf{g}}$, as the former is force per unit charge and the latter force per unit mass.

§0.2.2 Drawing Electric Fields

Here are the laws to obey when drawing electric fields

1. Field lines **never cross**
2. Field lines go from **positive** to **negative**
3. In electrostatics, when reaching a conductor, the field lines meet **perpendicular** to the surface.
4. A set charge will have a proportional number of 'field lines' (flux).
5. The density of the field lines is indicative of the field strength.

Example 0.21 A positively charged conductive sphere is placed above an earthed conductive surface. Draw the field lines.



Note that if there is a field **inside** a conductor, then there is a current.

§0.2.3 Coulomb's Law

> *The force exerted on two point charges is proportional to the product of the two charges and inversely proportional to the square of their separation.*

For two charges q_1 and q_2 a distance r away from each other, the electrical force is given by

Formula 0.22 (Coulomb's Law)

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$$

where ε_0 is the permittivity of free space ($= 8.85 \times 10^{-12}\ \mathrm{C^2\,N^{-1}\,m^{-2}}$).

Remark 0.23 Sometimes we may replace $\frac{1}{4\pi\varepsilon_0}$ with a k for convenience.

The electrical forces exerted on both charges are equal and in opposite directions, hence they form a Newton's Third Law pair, much like gravitational forces.
If q_1 and q_2 are both positive or both negative, F is positive $\Rightarrow$ **repulsive**.
If q_1 and q_2 are opposite charges, F is negative $\Rightarrow$ **attractive**.
We can use Coulomb's law to obtain a formula for the electric field **strength**, which is the magnitude of the electric field ($i.e$ $E = |\vec{\mathbf{E}}|$). Using $E = \frac{F}{q}$

Formula 0.24 (Electric Field Strength)

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Example 0.25 Two charges of -5 nC and -2 nC are 0.1 m away from each other. Find the distance from the -5 nC charge, where the electric field is zero.

Solution. Let x be the distance from the -5 nC where the electric field is zero. Then we have

$$\begin{aligned}
E_1(x) &= E_2(0.1-x) \\
\frac{-5}{4\pi\varepsilon_0 x^2} &= \frac{-2}{4\pi\varepsilon_0(0.1-x)^2} \\
\left(\frac{0.1-x}{x}\right) &= \sqrt{\frac{2}{5}} \\
\therefore \quad x &= \frac{0.1}{1+\sqrt{\frac{2}{5}}} \approx \underline{0.061 \text{ m}}
\end{aligned}$$

□

§0.2.4 Electric Potential and EPE

Definition 0.26 (Electric Potential Energy) — The work done in moving a charge from infinity to a point in an electric field.

Definition 0.27 (Electric Potential) — The work done per unit positive charge in moving a charge from infinity to a point in an electric field.

Remark 0.28 Unfortunately, those are just the typical, hand-wavy definitions required for A-Level. Below, I present a more mathematically rigorous argument.

The definition of potential energy U is $U(X) = -\int_{x_0}^{X} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{x}}$.

$$\begin{aligned}
\text{work done } &= -\int_{\infty}^{R} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} \\
&= -\int_{\infty}^{R} \frac{Qq}{4\pi\varepsilon_0 r^2} \mathrm{d}r \\
&= -\frac{Qq}{4\pi\varepsilon} \int_{\infty}^{R} \frac{\mathrm{d}r}{r^2} \\
&= -\frac{Qq}{4\pi\varepsilon} \left[-\frac{1}{r}\right]_{\infty}^{R} \\
&= \frac{Qq}{4\pi\varepsilon_0 R}
\end{aligned}$$

Now, the electric potential is simply the potential energy per unit charge of the 'test particle'.

Formula 0.29 (Electric Potential Energy)

$$E = \frac{Qq}{4\pi\varepsilon_0 r}$$

Formula 0.30 (Electric Potential)

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

where electric potential V has units of volts.
Electric potential is a **scalar field** associated with a charge Q at a distance r away.
An interesting corollary from the two previous formulas is that the electric field ($\vec{\mathbf{E}}$) can be defined as the **vector gradient** of the potential field.

Corollary 0.31

$$\vec{\mathbf{E}} = -\vec{\nabla} V$$

In a **uniform** electric field, the voltage increases linearly.

Formula 0.32 (Uniform Electric Field)

$$E = \frac{V}{d}$$

Remark 0.33 This formula is most common with two charged plates distance d apart.

Example 0.34 How fast would two deuterium nuclei be flying apart if they were originally 10^{-15} m apart (and initially stationary)?

§0.2.5 E-fields using vectors

We can express electric force $\vec{\mathbf{F}}$ and electric field $\vec{\mathbf{E}}$ in **vector form**.

$$\vec{\mathbf{F}} = \frac{Qq}{4\pi\varepsilon_0 r^2}\,\hat{\mathbf{r}} = \frac{Qq}{4\pi\varepsilon_0 r^3}\,\vec{\mathbf{r}}$$

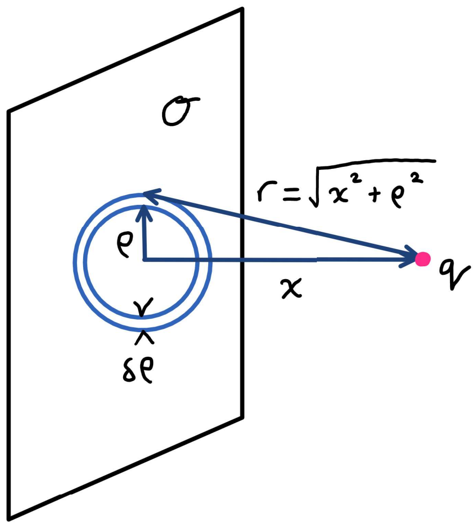
$$\vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{\mathbf{r}} = \frac{Q}{4\pi\varepsilon_0 r^3}\,\vec{\mathbf{r}}$$

§0.2.6 Parallel Plates

We consider a question:

> *Given a large, uniformly charged plate, with surface charge density σ, what is the voltage at a point above the centre of the plate, a distance x away?*

We have the following setup:



The quantity of charge δQ at radius ρ from the centre of the plate will be equal to the area of the thin ring[3] multiplied by **surface charge density** σ.

$$\delta q \approx \sigma \cdot \delta A = \sigma \cdot 2\pi\rho \cdot \delta\rho$$

But, we want the voltage at a distance r away, or x away from the centre where the charge at this point is q. We know that voltage is the **total** electric potential energy divided by the charge. Hence,

$$\begin{aligned} V \approx \frac{1}{q}\sum \delta U_E &= \frac{1}{q}\sum \frac{q \cdot \delta Q}{4\pi\varepsilon_0 r} \\ &= \sum \frac{2\pi\sigma\rho \cdot \delta Q}{4\pi\varepsilon_0 r} \\ &= \sum \frac{\sigma\rho \cdot \delta\rho}{2\varepsilon_0\sqrt{x^2+\rho^2}} \end{aligned}$$

Convert the sum to integral form for ρ from 0 up to the full radius of the plate ρ_{max}.

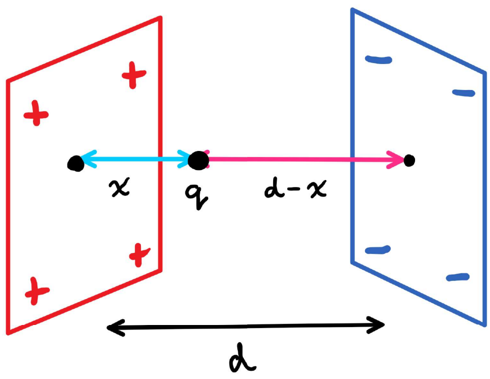
$$V = \frac{\sigma}{2\varepsilon_0}\int_0^{\rho_{max}} \frac{\rho}{\sqrt{x^2+\rho^2}}\,\mathrm{d}\rho$$

To evaluate this integral, we can use the substitution $u = \sqrt{x^2+\rho^2} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}\rho} = 2\rho$.

$$\begin{aligned} \therefore\ V =& \frac{\sigma}{2\varepsilon_0}\int_{u=x^2}^{u=x^2+\rho_{max}^2} \frac{\mathrm{d}u}{2\sqrt{u}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{u}\right]_{x^2}^{x^2+\rho_{max}^2} \\ &= \frac{\sigma}{2\varepsilon_0}\left(\sqrt{x^2+\rho_{max}^2} - x\right) \end{aligned}$$

Now, what if we place another **oppositely charged** plate on the other side of the point, with equal charge density σ, giving us the following setup.

[3]Note that we have used rings to sum the area of the plate to make the geometry and our calculations easier - it doesn't really make a difference.



We can now find the voltage V at a general position between these two parallel plates.

$$\begin{aligned} V &= V(x) - V(x-d) \\ &= \frac{\sigma}{2\varepsilon_0}\left(\sqrt{x^2+\rho^2} - x - \sqrt{(x-d)^2+\rho^2} - (x-d)^2\right) \end{aligned}$$

But, provided that ρ_{max} is considerably larger than x and d, $\sqrt{x^2+\rho^2} \approx \sqrt{\rho^2+(x-d)^2} \approx \rho$.

$$V = \frac{\sigma}{2\varepsilon_0}(d-2x)$$

This equation states that the voltage between oppositely charged plates varies **linearly** with distance.

$$\begin{aligned} &\text{at } x = 0: && V = \tfrac{\sigma d}{2\varepsilon_0} \\ &\text{at } x = \tfrac{d}{2}: && V = 0 \\ &\text{at } x = d: && V = -\tfrac{\sigma d}{2\varepsilon_0} \end{aligned}$$

For the **potential difference** between two parallel plates ΔV, we have $\Delta V = V(0) - V(d) = \frac{\sigma d}{\varepsilon_0}$, but recalling that $\sigma = \frac{Q}{A}$, where Q is the charge of the plates and A is the area of the plates, we obtain the following useful formula.

Formula 0.35 (Potential difference between parallel plates)

$$\Delta V = \frac{Qd}{A\varepsilon_0}$$

A natural question to ask now is what about the **electric field** $\vec{\mathbf{E}}$ between parallel plates? Recall from 0.31 that $\vec{\mathbf{E}}$ is just the **gradient** of voltage.

$$\begin{aligned} \vec{\mathbf{E}} = -\vec{\nabla} V &= -\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot V \\ &= -\frac{\sigma}{2\varepsilon_0}\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}(d-2x) \\ &= -\frac{\sigma}{2\varepsilon_0}\begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\sigma}{\varepsilon_0} \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Formula 0.36 (Electric field strength of parallel plates)

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$$

The $\vec{\mathbf{E}}$-field between parallel plates is constant in strength and uni-directional; a **uniform field**! This is essentially a proof of 0.32.

§0.2.7 Gauss' Law

The total flux (number of electric field lines) through any closed surface is proportional to the amount of charge enclosed within said surface.

Formula 0.37 (Gauss' Law)

$$\int \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}} = \frac{Q_{enc}}{\varepsilon}$$

Remark 0.38 $\mathrm{d}\vec{\mathbf{A}}$ is an **area vector** with direction **normal** to the surface. ε is the electrical permittivity, equal to ε_0 in a vacuum.

Depending on the problem, we choose a 'surface' which makes calculation the most easy.

We can use Gauss' Law to show the results for the electric field strength of a spherical or point charge and parallel plates. I leave this as an excercise for the reader. Note that the electric fields are uniform in these situations.

Example 0.39 Use Gauss' Law to find the electric field strength a distance r from a charged wire.

Solution.

$$\frac{Q_{enc}}{\varepsilon} = \int E \cdot \mathrm{d}A = E \cdot A_{total}$$

We can model our surface with the geometry of a cylinder, with our (straight-line) wire within this surface. Let's give the wire length l and we have the distance from the wire to the surface boundary r.

$$\therefore \quad \frac{Q_{enc}}{\varepsilon} = E \cdot 2\pi r l$$
$$\Rightarrow E = \frac{Q}{\varepsilon \cdot 2\pi r l}$$

But, since the total charge enclosed and the length of the wire are constants, we can make the substitution that $\frac{Q}{\varepsilon} = \lambda$, giving the nicer looking expression:

$$E = \frac{\lambda}{\varepsilon \cdot 2\pi r}$$

□